ICM-9110-15 Is the Optimal ω Best for the SOR Iteration Method?, M. Eiermann and R.S. Varga, Linear Algebra and Its Applications (to appear).

ABSTRACT: The successive overrelaxation (SOR) iterative method for linear systems is well understood if the associated Jacobi matrix B is consistently ordered and weakly cyclic of index 2. If, in addition, B^2 has only nonnegative eigenvalues and if $\rho(B)$, the spectral radius of B, is strictly less than unity, then by D. M. Young's classical theorem [8], the optimal relaxation parameter for the SOR method is given by

$$\omega_b := \frac{2}{1+\sqrt{1-\rho^2(B)}}.$$

Young derived this result assuming that $(*)$ $\sigma(B^2) \subset [0, \beta^2]$ (with $\beta = \rho(B)$) is the only information available about the spectrum $\sigma(B^2)$ of B^2. It is also well-known [6] that no polynomial acceleration can improve the asymptotic rate of convergence of the SOR scheme if the optimal relaxation parameter has been selected.

The recent claim by J. Dancis [1, p. 819] "that a smaller average spectral radius can be achieved by using a polynomial acceleration together with a suboptimal relaxation factor ($\omega < \omega_b$)" therefore comes as a surprise. A closer look however reveals that this improvement can only be achieved if more profound information on $\sigma(B^2)$, of the form $(**)$ $\sigma(B^2) \subset [0, \gamma^2] \cup \{\beta^2\}$ (with $\gamma < \beta$), is at hand. We shall show that no polynomial acceleration of the SOR method (for *any* real ω) is asymptotically faster than the SOR scheme with $\omega = \omega_b$ under the assumption $(*)$, thereby answering the question in the title of this paper in the affirmative, as well as solving an old related conjecture of D.M. Young [9, p. 376].