ICM-9205-33 Some Numerical Results on Best Uniform Polynomial Amos J. Carpenter and R.S. Varga, Approximation of x^α on $[0,1]$, USSR-US Conference on Approximation Theory, (to appear).

ABSTRACT: Let α be a positive number, and let $E_n(x^\alpha;[0,1])$ denote the error of best uniform approximation to x^α, by polynomials of degree at most n, on the interval $[0,1]$. The Russian mathematician S. N. Bernstein established the existence of a nonnegative constant $\beta(\alpha)$ such that

$$\beta(\alpha) := \lim_{n\to\infty} (2n)^{2\alpha} E_n(x^\alpha;[0,1]) \qquad (\alpha > 0).$$

In addition, Bernstein showed that

$$\beta(\alpha) < \frac{\Gamma(2\alpha)|\sin(\pi\alpha)|}{\pi} \qquad (\alpha > 0),$$

and that

$$\frac{\Gamma(2\alpha)|\sin(\pi\alpha)|}{\pi}\left(1 - \frac{1}{2\alpha - 1}\right) < \beta(\alpha) \qquad (\alpha > \frac{1}{2}),$$

so that the asymptotic behavior of $\beta(\alpha)$ is thus known when $\alpha \to \infty$.

Still, the problem of trying to determine $\beta(\alpha)$ more precisely, for all $\alpha > 0$, is intriguing. To this end, we have rigorously determined the numbers $\{E_n(x^\alpha;[0,1])\}_{n=1}^{40}$ for thirteen values of α, where these numbers were calculated with a precision of at least 200 significant digits. For each of these thirteen values of α, Richardson's extrapolation was applied to the products $\left\{(2n)^{2\alpha} E_n(x^\alpha;[0,1])\right\}_{n=1}^{40}$ to obtain estimates of $\beta(\alpha)$ to approximately 40 decimal places. Included are graphs of the points $(\alpha, \beta(\alpha))$ for the thirteen values of α that we considered.