LIPSCHITZ-TYPE BOUNDS FOR THE MAP $A \to |A|$ ON $\mathcal{L}(\mathcal{H})$.

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Abstract

It is well known that the absolute value map on the self adjoint operators on an infinite dimensional Hilbert spaces is not Lipschitz continuous, although Lipschitz continuity holds on certain subsets of operators. In this note, we provide an elementary proof that the absolute value map is Lipschitz continuous on the set of all operators which are bounded below in norm by any fixed positive constant. Applications are indicated.

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Throughout this note, let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Denote by $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$ and $\mathcal{L}^{\text{P}}(\mathcal{H})$ the subalgebras of $\mathcal{L}(\mathcal{H})$ consisting of self adjoint and non-negative operators. Recall that a map $f : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is called *Lipschitz continuous* if there exists a constant M such that for every $A, B \in \mathcal{L}(\mathcal{H})$, $||f(A) - f(B)|| \leq M||A - B||$. In the text [RS], it was pointed out that it was unknown whether the absolute value mapping, $A \to |A| = \sqrt{A^*A}$, is Lipschitz continuous on $\mathcal{L}(\mathcal{H})$. In his 1973 paper [K1], Kato provided a counterexample showing that Lipschitz continuity of the absolute value map fails on the subalgebra of self adjoint operators, $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$. Since then other studies of the Lipschitz continuity of the absolute value map on self adjoint operators have been offered. Notably, some interesting counter examples were given in [M] and the Lipschitz continuity of the absolute value map on finite dimensional spaces was proved in [F]. Many other logarithmic Lipschitz bounds have been developed for various important functions on operators (cf. [AD], [F2], [K2]).

In [B1] and [B3], using perturbations of Frechet derivatives, it is shown that the absolute value map satisfies a bound of the form $|||A| - |B||| \leq a_1||A - B|| + a_2||A - B||^2 + \mathcal{O}(||A -$

$B||^3$) provided A is invertible and B is sufficiently close to A. In this note, we drop the invertibility requirements and the error terms $a_2||A - B||^2 + \mathcal{O}(||A - B||^3)$ in the bound. Specifically, we show that the absolute value map is Lipschitz continuous on the set of all operators which are bounded below in norm by any fixed positive constant. We apply our construction to provide an example which shows that there are subsets of $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$ which properly contain $\mathcal{L}^{\text{P}}(\mathcal{H})$ on which the absolute value map is norm Lipschitz continuous. In particular, this provides an example of a subset of $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$ containing operators of arbitrarily small norm on which the absolute value map is Lipschitz continuous. This, in turn, yields an important interpretation in terms of the logarithmic Lipschitz continuity property of the absolute value map which was established in [K1]. We begin by showing that the square root function is Lipschitz continuous on positive operators whose norms are bounded below by a fixed constant.

Definition. For any constant $R > 0$, define

$$B_R = \{A \in \mathcal{L}(\mathcal{H}) \ : \ ||A|| > R\} \quad \text{and} \quad B_R^+ = \{A \in \mathcal{L}^{\text{P}}(\mathcal{H}) \ : \ ||A|| > R\}.$$

Lemma 1. *Fix R with $0 < R < \frac{1}{16}$. The square root map is Lipschitz continuous on the set B_R^+. Specifically, whenever A and B are operators in $\mathcal{L}^{\text{P}}(\mathcal{H})$ with $R < ||B|| \leq ||A||$,*

$$(1) \qquad\qquad\qquad\qquad ||\sqrt{A} - \sqrt{B}|| \leq \frac{||A - B||}{\sqrt{R||A||}}.$$

Thus, for all $A, B \in B_R^+$,

$$(2) \qquad\qquad\qquad\qquad ||\sqrt{A} - \sqrt{B}|| \leq \frac{||A - B||}{R}.$$

Proof. Fix R with $0 < R < \frac{1}{16}$. Let $A, B \in B_R^+$ with $R < ||B|| \leq ||A||$ We consider separately the cases: (i) $\sqrt{R}||A|| > ||B||$ and (ii) $\sqrt{R}||A|| \leq ||B||$. In case (i), $\sqrt[4]{R}\sqrt{||A||} > \sqrt{||B||}$ and so

$$(1 - \sqrt[4]{R})\sqrt{||A||} < \sqrt{||A||} - \sqrt{||B||}.$$

since $R < \frac{1}{16}$ implies that $\sqrt{R} < (1 - \sqrt[4]{R})$, this yields:

$$||\sqrt{A} - \sqrt{B}|| \leq ||\sqrt{A}|| + ||\sqrt{B}|| = \frac{||A|| - ||B||}{\sqrt{||A||} - \sqrt{||B||}} \leq \frac{||A|| - ||B||}{\sqrt{R}\sqrt{||A||}} \leq \frac{||A - B||}{\sqrt{R}\sqrt{||A||}}.$$

Now assume $\sqrt{R}||A|| \leq ||B||$. That is, $\frac{||A||}{||B||} < \frac{1}{\sqrt{R}}$. Recall we can use a power series expansion to calculate the square root of any non-negative operator E which has norm less than 1. Specifically,

$$\sqrt{E} = \sum_{n=0}^{\infty} c_n (I - E)^n$$

where $\sum_{n=1}^{\infty} |c_n| \leq 1$ (cf. [RS]). With this in mind, we fix $\epsilon > 0$ and define $C = I - \frac{A}{(1+\epsilon)\|A\|}$ and $D = I - \frac{B}{(1+\epsilon)\|A\|}$. Then

$$\|C\| = 1 - \frac{1}{1+\epsilon} \quad \text{and} \quad \|D\| = 1 - \frac{\|B\|}{(1+\epsilon)\|A\|}.$$

Let $\Gamma = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then $\sigma(C) \cup \sigma(D)$ is contained inside Γ. It follows that for every natural number $m \geq 2$,

$$\|C^m - D^m\| = \|\frac{1}{2\pi i} \int_{\Gamma} w^m [(w-C)^{-1} - (w-D)^{-1}] dw\|$$

$$\leq \sup_{w \in \Gamma} \|(w-C)^{-1} - (w-D)^{-1}\|$$

$$\leq \sup_{w \in \Gamma} \|(w-C)^{-1}\|\|C-D\|\|(w-D)^{-1}\|$$

$$\leq \frac{1}{\text{dist}(\Gamma, \sigma(C))} \|C-D\| \frac{1}{\text{dist}(\Gamma, \sigma(D))}$$

$$= \frac{\|C-D\|}{(1-\|C\|)(1-\|D\|)}$$

$$= \frac{(1+\epsilon)^2 \|A\|\|C-D\|}{\|B\|}$$

$$\leq \frac{(1+\epsilon)^2 \|C-D\|}{\sqrt{R}}$$

Here we used the fact $\|E^{-1} - F^{-1}\| = \|E^{-1}(E-F)F^{-1}\| \leq \|E^{-1}\|\|E-F\|\|F^{-1}\|$ which holds for any invertible operators E and F, as well as the fact that for each $w \in \Gamma$, $\|(w-C)^{-1}\| = \frac{1}{\text{dist}(w, \sigma(C))}$.

Since power series convergence is absolute and the operators $I - C$ and $I - D$ are non-negative operators of norm less than 1, we conclude

$$\frac{1}{\sqrt{(1+\epsilon)\|A\|}} \|\sqrt{A} - \sqrt{B}\| = \|\sqrt{I-C} - \sqrt{I-D}\|$$

$$= \|\sum_{n=0}^{\infty} c_n (C^n - D^n)\|$$

$$\leq \sum_{n=1}^{\infty} |c_n|\|C^n - D^n\|$$

$$\leq \frac{(1+\epsilon)^2}{\sqrt{R}} \|C-D\| = \frac{(1+\epsilon)}{\sqrt{R}\|A\|} \|A-B\|.$$

Hence,

$$\|\sqrt{A} - \sqrt{B}\| \leq \frac{(\sqrt{1+\epsilon})^{\frac{3}{2}}}{\sqrt{R}\sqrt{\|A\|}} \|A-B\|.$$

Since the above inequality holds for any arbitrarily small positive ϵ, we have established inequality (1) of the lemma. Statement (2) of the lemma follows directly from (1) and the fact that $R < ||A||$. $\square$

The above lemma implies a Lipschitz continuity result for the absolute value function on $\mathcal{B}_R$. To see this, fix any operators A and B and observe

$$|||A|^2 - |B|^2|| \leq ||A^*|| ||A - B|| + ||A^* - B^*|| ||B|| = (||A|| + ||B||) ||A - B||.$$

The above bound is sharp as can be seen from the operators A and B which are zero except for $A_{(1,1)} = 2$ and $B_{(1,1)} = 1$. More importantly, the bound implies if $||B|| \leq ||A||$ then $|||A|^2 - |B|^2|| \leq 2||A|| ||A - B||$. We apply this in proving the following lemma.

Lemma 2. *Fix R with $0 < R < \frac{1}{4}$. The absolute value map $A \to |A| = \sqrt{A^*A}$ is Lipschitz continuous on the set B_R. In fact, for all $A, B \in B_R$,*

$$|||A| - |B||| \leq \frac{2||A - B||}{R}.$$

Proof. Let $A, B \in B_R$, with $R < ||B|| \leq ||A||$. Note $|A|^2$ and $|B|^2$ lie in $B_{R^2}^+$. Hence by combining equation (1) of Lemma 1 with the above calculation we see

$$|||A| - |B||| = ||\sqrt{|A|^2} - \sqrt{|B|^2}|| \leq \frac{1}{\sqrt{R^2}\sqrt{||A||^2}} |||A|^2 - |B|^2|| \leq \frac{2||A||}{R||A||} ||A - B||.$$

This establishes the lemma. $\square$

Since the above lemmas hold for arbitrarily small positive choices of R, we have actually proven:

Theorem. *Fix $R > 0$. The square root map is Lipschitz continuous on the set, B_R^+, of all non-negative operators with norm greater than R and the absolute value map is Lipschitz continuous on the set, B_R, of all operators with norm greater than R. That is, there are constants M_R and N_R such that*

$$||\sqrt{A} - \sqrt{B}|| \leq M_R||A - B|| \quad \forall A, B \in B_R^+$$

and

$$|||A| - |B||| \leq N_R||A - B|| \quad \forall A, B \in B_R.$$

Using the above theorem we can now construct subsets $\mathcal{I}_R$ of $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$ which properly contain $\mathcal{L}^{\text{P}}(\mathcal{H})$ on which the absolute value map is norm Lipschitz continuous.

Example. Let $R > 0$ and define

$$\mathcal{I}_R = \{A \in \mathcal{L}^{\text{S.A.}}(\mathcal{H}) \ : \ \sigma(A) \cap (-R, 0) = \emptyset\}$$

where $\sigma(A)$ denotes the spectrum of A. Note that it suffices to establish Lipschitz continuity for all A and B in $\mathcal{I}_R$ such that $||A - B|| \leq \frac{R}{2}$. Indeed, if we establish a Lipschitz constant $K = K_R > 0$ in this case then we can extend to all of $\mathcal{I}_R$ by scaling. That is, for any $C, D \in \mathcal{I}_R$, let $\alpha = ||C - D||$. If $\alpha > \frac{R}{2}$ we can multiply C and D by $\beta = \frac{R}{2\alpha}$. Then

$$|||C| - |D||| = \frac{1}{\beta}|||\beta C| - |\beta D||| \leq \frac{K}{\beta}||\beta C - \beta D|| = K||C - D||.$$

Hence we assume $A, B \in \mathcal{I}_R$ with $||A - B|| \leq \frac{R}{2}$. In this case, either both A and B are both non-negative (and the Lipschitz constant 1 works trivially) or $||A||, ||B|| \geq R$ (and the Lipschitz constant given by Theorem 1 applies).

It remains only to show why these are the only two possibilities. Let $A, B \in \mathcal{I}_R$ with $||A - B|| \leq \frac{R}{2}$ and suppose A is not a non-negative operator. Since A is self adjoint, this could only happen if $\sigma(A) \cap (-\infty, 0) \neq \emptyset$. But by hypothesis, $\sigma(A) \cap (-R, 0) = \emptyset$ and so there must exist a vector unit $x \in \mathcal{H}$ such that $< Ax, x > \leq -R$. But,

$$| < (B - A)x, x > | \leq ||A - B|| \leq \frac{R}{2}$$

This implies $< Bx, x > \leq -\frac{R}{2}$. Of course, this implies that B is not a non-negative operator and since $B \in \mathcal{I}_R$ we must have $||B|| \geq R$.

Remark. Note that the above example does more than show that Lipschitz continuity of the absolute value map can hold on proper supersets of the positive operators. It provides an interesting insight into logarithmic Lipschitz bound on the absolute value map on $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$ which was established in [K1]. More precisely, it was proved in [K1] that for all $A, B \in \mathcal{L}^{\text{S.A.}}(\mathcal{H})$,

$$|| |A| - |B| || \leq [\frac{4}{\pi} + \frac{2}{\pi} \log \frac{||A|| + ||B||}{||A - B||}] ||A - B||.$$

We note that on the subsets $\mathcal{I}_R$ the absolute value is Lipschitz continuous even though the above logarithmic bound function tends to infinity. This suggests that investigations of logarithmic Lipschitz bounds for the absolute map on $\mathcal{L}^{\text{S.A.}}(\mathcal{H})$ could be a very interesting line of inquiry.

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